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Similar Solutions for Three-Dimensional Laminar Compressible Boundary Layers

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Nomenclature

e_1, e_2	= metric coefficients
f, g	= similar functions
h	= enthalpy
H	= stagnation enthalpy, $h + \frac{1}{2}(u^2 + w^2)$
K_1	= curvature parameter, $(2\xi/e_1)(\partial e_1/\partial \zeta)$
K_2	= dilatation parameter, $(2\xi/e_2)(\partial e_2/\partial \xi)$
K^*	= $(\partial/\partial\zeta^*)(\beta^*/rZ)$
M	= Mach number
m	= $[(\gamma - 1)/2]M_e^2$
$\hat{m}$	= $1 + [(\gamma - 1)/2]M_e^2$
N	= $\rho\mu/\rho_0\mu_0$
p	= pressure
P_r	= Prandtl number
r	= body radius
T	= temperature
u, v, w	= velocity components
x, y, z	= curvilinear orthogonal coordinates
α	= $2\xi(dZ/d\zeta)$
β	= pressure gradient parameter, $(2\xi/u_e)(\partial u_e/\partial\xi)\hat{m}$
β^*	= $-(2\xi/\rho_e u_e^2)(\partial p/\partial\zeta)$
ξ, η, ζ	= transformed coordinates
ψ, ϕ	= stream functions
ρ	= density
μ	= dynamic viscosity
θ	= total enthalpy ratio, H/He
ω	= spinning rate

Subscripts

0	= reference state
∞	= freestream condition
e	= outer-edge condition

Superscript

$(\ )'$	= derivative with respect to η

THE three-dimensional boundary layer is characterized by crossflow generation and streamline deviation within the boundary layer. An adequate mathematical description of these phenomena entails the consideration of several additional nonlinear terms in the governing equations which are absent in the corresponding two-dimensional or axisymmetric equations. One approach to this problem is to invoke the concept of the streamline coordinate system for which one of the coordinates is set to be coincident with the local outer-edge streamline projected in the tangent plane. By so doing, the perturbation technique in treating the crossflow effect can be justified for a wide variety of practical applications, and because of the presence of only one external flow component, the construction of a single similarity parameter can be achieved without introducing any restrictive assumptions. The purpose of this note is to discuss a certain class of similar solutions for three-dimensional laminar compressible boundary-layer flows on the basis of the streamline coordinate system and by employing a set of simple transformation variables. Neglected herein are the real gas effects associated with hypersonic flight, flow separation, and laminar-to-turbulent transition phenomena, and the analytical procedures for the determination of surface streamlines in accordance with the theory of differential geometry.

Similar to the consideration by Beckwith,[1] we select a three-dimensional orthogonal streamline coordinate system (x,y,z) with corresponding velocity components u tangent to the external streamline, v normal to the body surface, and w, the crossflow velocity component, normal to u in the tangent plane. The length elements are

$$ds = e_1(x, z)dx \qquad dy = dy \qquad dn = e_2(x, z)dz \tag{1}$$

and at the outer edge of the boundary layer, $u = u_e$ and $w = w_e = 0$. The governing equations for a three-dimensional laminar compressible boundary layer may then be written as:

Continuity

$$\frac{1}{e_1}\frac{\partial}{\partial x}(\rho u e_2) + \frac{\partial}{\partial y}(\rho v e_2) + \frac{1}{e_1}\frac{\partial}{\partial z}(\rho w e_1) = 0 \tag{2}$$

x Momentum

$$\frac{\rho u}{e_1}\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \frac{\rho w}{e_2}\frac{\partial u}{\partial z} + \frac{\rho u w}{e_1 e_2}\frac{\partial e_1}{\partial z} - \frac{\rho w^2}{e_1 e_2}\frac{\partial e_2}{\partial x} = -\frac{1}{e_1}\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{3}$$

y Momentum

$$\partial p/\partial y = 0 \tag{4}$$

z Momentum

$$\frac{\rho u}{e_1}\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \frac{\rho w}{e_2}\frac{\partial w}{\partial z} + \frac{\rho u w}{e_1 e_2}\frac{\partial e_2}{\partial x} - \frac{\rho u^2}{e_1 e_2}\frac{\partial e_1}{\partial z} = -\frac{1}{e_2}\frac{\partial p}{\partial z} + \frac{\partial}{\partial y}\left(\mu\frac{\partial w}{\partial y}\right) \tag{5}$$

Energy

$$\frac{\rho u}{e_1}\frac{\partial H}{\partial x} + \rho v\frac{\partial H}{\partial y} + \frac{\rho w}{e_2}\frac{\partial H}{\partial z} = \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial H}{\partial y} + \frac{1 - P_r}{P_r}\frac{\partial h}{\partial y}\right)\right] \tag{6}$$

From the continuity equation, we define two stream functions ψ and ϕ as follows:

$$\rho u e_2 = \frac{\partial\psi}{\partial y} \qquad \rho v e_2 = \left(-\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial z}\right)\frac{1}{e_1} \qquad \rho w e_1 = \frac{\partial\phi}{\partial y} \tag{7}$$

We now introduce the transformation variables

$$\xi = \int_0^x \rho_0\mu_0 e_1 dx = \int_0^s \rho_0\mu_0 ds$$
$$\eta = \left(\frac{u_e}{2\xi}\right)^{1/2}\int_0^y \rho dy \tag{8}$$
$$\zeta = \int_0^z \rho_0\mu_0 e_2 dz = \int_0^n \rho_0\mu_0 dn$$

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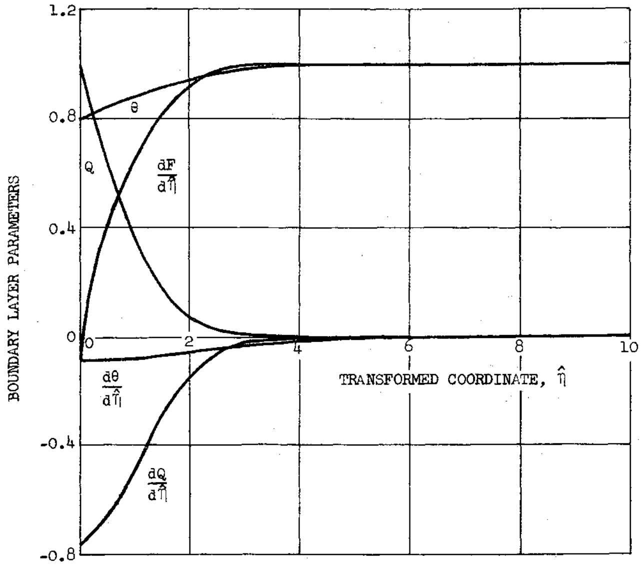

Fig. 1 Boundary-layer parameters vs transformed coordinate η for spin parameter $\omega^2\lambda = 0.005$.

and define related functions $f(\xi, \eta, \zeta)$ and $g(\xi, \eta, \zeta)$ as follows:

$$\psi = (2\xi u_e)^{1/2} e_2 f(\xi, \eta, \zeta) \qquad \phi = (2\xi u_e)^{1/2} e_1 g(\xi, \eta, \zeta) \qquad (9)$$

from which we obtain

$$u = u_e(\partial f/\partial \eta) \qquad w = u_e(\partial g/\partial \eta) \qquad (10)$$

Substituting these transformation variables into Eqs. (3–6) and noting that at the outer edge of the boundary layer

$$\frac{\rho_e u_e}{e_1}\frac{\partial u_e}{\partial x} = -\frac{1}{e_1}\frac{\partial p}{\partial x}$$
$$\frac{1}{e_1}\frac{\partial e_1}{\partial z} = -\frac{1}{u_e}\frac{\partial u_e}{\partial z} = \frac{1}{\rho_e u_e^2}\frac{\partial p}{\partial z} \qquad (11)$$

and across the boundary layer

$$\frac{\rho_e}{\rho} = \frac{T}{T_e} = \theta \hat{m} - m\left[\left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial g}{\partial \eta}\right)^2\right] \qquad (12)$$

the following equations are obtained:

$$\frac{\partial}{\partial \eta}\left(N\frac{\partial^2 f}{\partial \eta^2}\right) + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f \frac{\partial^2 f}{\partial \eta^2} + \beta\left[\theta - \left(\frac{\partial f}{\partial \eta}\right)^2\right] + \left(K_2 - \beta\frac{m}{\hat{m}}\right)\left(\frac{\partial g}{\partial \eta}\right)^2 + \frac{K_1}{2} g \frac{\partial^2 f}{\partial \eta^2} + 2\xi\left(\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta}\right) + 2\xi\left(\frac{\partial g}{\partial \zeta}\frac{\partial^2 f}{\partial \eta^2} - \frac{\partial g}{\partial \eta}\frac{\partial^2 f}{\partial \zeta \partial \eta}\right) = 0 \qquad (13)$$

$$\frac{\partial}{\partial \eta}\left(N\frac{\partial^2 g}{\partial \eta^2}\right) + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f \frac{\partial^2 g}{\partial \eta^2} + K_1\left[\left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial g}{\partial \eta}\right)^2 + \frac{g}{2}\frac{\partial^2 g}{\partial \eta^2}\right] + \beta^*\left(\frac{\rho_e}{\rho}\right) - \left(K_2 + \frac{\beta}{\hat{m}}\right)\left(\frac{\partial f}{\partial \eta}\frac{\partial g}{\partial \eta}\right) + 2\xi\left(\frac{\partial f}{\partial \xi}\frac{\partial^2 y}{\partial \eta^2} - \frac{\partial f}{\partial \eta}\frac{\partial^2 g}{\partial \xi \partial \eta}\right) + 2\xi\left(\frac{\partial g}{\partial \zeta}\frac{\partial^2 g}{\partial \eta^2} - \frac{\partial g}{\partial \eta}\frac{\partial^2 g}{\partial \zeta \partial \eta}\right) = 0 \qquad (14)$$

$$\frac{1}{P_r}\frac{\partial}{\partial \eta}\left(N\frac{\partial \theta}{\partial \eta}\right) - \frac{1 - P_r}{P_r}\frac{m}{\hat{m}}\frac{\partial}{\partial \eta}\left\{N\frac{\partial}{\partial \eta}\left[\left(\frac{\partial f}{\partial \eta}\right)^2 + \left(\frac{\partial g}{\partial \eta}\right)^2\right]\right\} + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f \frac{\partial \theta}{\partial \eta} + \frac{K_1}{2} g \frac{\partial \theta}{\partial \eta} + 2\xi\left(\frac{\partial f}{\partial \xi}\frac{\partial \theta}{\partial \eta} - \frac{\partial f}{\partial \eta}\frac{\partial \theta}{\partial \xi}\right) + 2\xi\left(\frac{\partial g}{\partial \zeta}\frac{\partial \theta}{\partial \eta} - \frac{\partial g}{\partial \eta}\frac{\partial \theta}{\partial \zeta}\right) = 0 \qquad (15)$$

where appropriate boundary conditions must be prescribed for a given problem. It should be remarked that the parameter β^*, which is equal to $-K_1$, is purposely retained in Eq. (14) since, for certain special three-dimensional viscous flow problems, the crossflow effect can best be demonstrated by first taking termwise differentiation of Eq. (5) with respect to ζ, in which case $\partial\beta^*/\partial\zeta$ may be different from $-\partial K_1/\partial\zeta$.

We are at present concerned with a class of similar solutions wherein $g(\xi, \eta, \zeta) = Z(\zeta)\hat{g}(\xi, \eta)$, with Z and $\partial Z/\partial\zeta$ being known values along any given streamline under consideration. In accordance with the usual requirements for similar (or locally similar) flows, we assume that f, $\hat{g}$, and θ are primarily dependent on η. Then, Eqs. (11–13) become

$$f''' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) ff'' + \beta(\theta - f'^2) + Z^2\left(K_2 - \beta\frac{m}{\hat{m}}\right)\hat{g}'^2 + \left(\alpha + \frac{1}{2}K_1 Z\right)\hat{g}f'' = 0 \qquad (16)$$

$$\hat{g}''' + \left(1 + \frac{\beta}{2m} + K_2\right) f\hat{g}'' + \frac{K_1\hat{m}}{Z}\left[f'^2 - \theta + Z^2 \times \left(\hat{g}'^2 + \frac{\hat{g}\hat{g}''}{2\hat{m}}\right)\right] - \left(K_2 + \frac{\beta}{\hat{m}}\right)f'\hat{g}' + \alpha(\hat{g}\hat{g}'' - \hat{g}'^2) = 0 \qquad (17)$$

$$\frac{1}{P_r}\theta'' - \frac{1 - P_r}{P_r}\frac{m}{\hat{m}}(f'^2 + Z^2\hat{g}'^2)' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f\theta' + \left(\alpha + \frac{1}{2}K_1 Z\right)\hat{g}\theta' = 0 \qquad (18)$$

where $N = 1$, and α, β, m, $\hat{m}$, P_r, K_1, K_2, and Z are assumed to be constant, at least within the region in which the flow can be regarded as locally similar. Although the foregoing high-order, nonlinear generalized equations for similar flows are still difficult to solve, they are amenable to numerical treatments with the aid of a high-speed digital computer. Flow characteristics for a large number of viscous flow problems can then be determined if proper values are assigned to the constants and appropriate boundary conditions are prescribed. A few interesting special cases are exemplified below.

For the laminar flows in the stagnation region of a spinning axisymmetric blunt-nosed body at zero yaw,[2–4] the following values are assigned to the constants: $e_1 = 1$, $e_2 = r(x)$, $K_1 = 0$, $K_2 = 2$, $\alpha = 0$, $\beta = 2$, $m = 0$, and $\hat{m} = 1$. Furthermore, we may assume $Z\hat{g}' = 2\lambda^{1/2}\omega Q(\eta)$, where ω is the spinning rate and $\lambda = \frac{1}{2}(r/u_e)^2$, $f = \frac{1}{2}F$ and $(\ )' = 2(d/d\hat{\eta})(\ )$. Then, Eqs. (16–18) have the form

$$\frac{d^3F}{d\hat{\eta}^3} + F\frac{d^2F}{d\hat{\eta}^2} + \frac{1}{2}\left[\theta - \left(\frac{dF}{d\hat{\eta}}\right)^2\right] + \omega^2\lambda Q^2 = 0 \qquad (19)$$

$$\frac{d^2Q}{d\hat{\eta}^2} + F\frac{dQ}{d\hat{\eta}} - \frac{dF}{d\hat{\eta}}Q = 0 \qquad (20)$$

$$\frac{1}{P_r}\frac{d^2\theta}{d\hat{\eta}^2} + F\frac{d\theta}{d\hat{\eta}} = 0 \qquad (21)$$

with boundary conditions

$$F(0) = \frac{dF(0)}{d\hat{\eta}} = Q(\infty) = 0$$

$$\frac{dF(\infty)}{d\hat{\eta}} = Q(0) = \theta(\infty) = 1 \qquad \theta(0) = \frac{h_w}{H_e}$$

These equations are identical with the zero-order equations for the spinning-body problem obtained from employing the usual Mangler and Lees-Dorodnitsyn transformations.[4] For $\theta(0) = 0.8$, $P_r = 0.722$, $M_\infty = 5$, and $\lambda\omega^2 = 0.005$, which approximately corresponds to a spinning rate of $\omega = 1700$ rpm, the Runge-Kutta scheme was utilized to solve the foregoing differential equations. The calculated results of various functions are depicted in Fig. 1, and the heat-transfer ratio is found to be

$$\dot{q}_w/(\dot{q}_w)_{\omega=0} = 0.08025/0.08027 \approx 1$$

which indicates negligible spinning effect on convective heat transfer for flows in the stagnation-point region under hypersonic flight conditions. This result was predicted by Scala and Workman,[3] and experimentally verified by Whitesel.[5]

Another interesting case is that concerning small but finite crossflow for which the higher-order terms in $\hat{g}$, α, K_1, and their lateral derivative $\partial/\partial\zeta$ are neglected. Then, Eqs. (16–18) become (for $P_r = 1$)

$$f''' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) ff'' + \beta(\theta - f'^2) = 0 \quad (22)$$

$$\hat{g}''' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f\hat{g}'' + K_1\hat{m}(f'^2 - \theta) - \left(K_2 + \frac{\beta}{\hat{m}}\right) f'g' = 0 \quad (23)$$

$$\theta'' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) f\theta' = 0 \quad (24)$$

If we multiply Eq. (23) by e_2 and substitute $(e_2\hat{g})' = V(\eta)G(\xi)$ therein, we obtain

$$V'' + \left(1 + \frac{\beta}{2\hat{m}} + K_2\right) fV' - \left(\frac{\beta}{\hat{m}} + \frac{2\xi}{G}\frac{dG}{d\xi}\right) f'V = \frac{\hat{m}K_1e_2}{G}(\theta - f'^2) \quad (25)$$

It is noted that Eqs. (22, 24, and 25) are essentially those obtained by Beckwith[1] except for the coefficient constants.

Finally, for the case of similar flows in the plane of symmetry of an inclined axisymmetric body with zero streamwise pressure gradient and insulated walls, the following conditions prevail: $e_1 = 1$, $e_2 = r(x)$, $K_1 = 0$, $\beta = 0$, and $\partial g/\partial\eta = 0$. In order to transform the resulting equations into a familiar form, we first differentiate Eq. (14) with respect to ζ^*, which is defined as $r\zeta^* = \zeta$. Then, with the aid of the following definitions:

$$\hat{F} = (1 + K_2)f \qquad \hat{G} = \frac{\partial g}{\partial\zeta^*} \qquad K^* = \frac{\partial}{\partial\zeta^*}\left(\frac{\beta^*}{rZ}\right)$$

$$C_1 = \frac{2\xi}{r} \qquad C_2 = \frac{2\xi}{r}(1 + K_2)^{-1}$$

we obtain

$$\hat{F}''' + (\hat{F} + C_1\hat{G})\hat{F}'' = 0 \quad (26)$$

$$\hat{G}''' + (\hat{F} + C_1\hat{G})\hat{G}'' + K^*(\rho_e/\rho) - C_1\hat{G}'^2 - C_2\hat{F}\hat{G}' = 0 \quad (27)$$

which are the governing equations for supersonic flows in the plane of symmetry of a yawed cone with insulated surface.[6]

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Large Amplitude Vibration of Buckled Beams and Rectangular Plates

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Nomenclature

a, b, h = plate width, length, and thickness (x, y, z directions), respectively
r = a/b, plate aspect ratio
t = time
u, v, w = displacements in the x, y, z directions, respectively
D = plate flexural rigidity, $Eh^3/12(1 - \nu^2)$
EI = beam flexural rigidity
F = stress function
ρ = mass density
ν = Poisson's ratio

Introduction

In recent years, a number of investigations of the large amplitude vibration of beams[1–4] and flat rectangular plates[5–8] have been reported in which the ends of the beams and the edges of the plates have been assumed to remain a fixed distance apart during vibration. In particular, Burgreen[2] has considered the free vibration of a simply supported beam that has been given an initial end displacement, and the author[8] has considered free and forced vibration of simply supported and clamped beams and rectangular plates for which initial end and edge displacements have been prescribed. In both reports, a one-degree-of-freedom representation of the equations of motion is used. Results are obtained for edge displacements in the postbuckling as well as the prebuckling region. In the case of forced motion, however, the results were restricted to symmetrical motion about the flat position of the beam or plate. For the buckled beam or plate, it is also possible to have vibration about the static buckled position. This has been discussed for free vibration in the forementioned reports, and it is the purpose of the following remarks to extend the discussion to a case of forced motion.

Equations of Motion

The differential equation of motion for a beam of unit width is

$$\rho hw_{,tt} + (EIw_{,yy})_{,yy} - \frac{Eh}{b}\left[v_0 + \frac{1}{2}\int_0^b (w_{,y})^2 dy\right] w_{,yy} = P(y, t) \quad (1)$$

where v_0 represents an initial axial displacement measured from the unstressed state. For a plate, the dynamic von Kármán equations are

$$\nabla^4 F = E(w_{,xy}{}^2 - w_{,xx}w_{,yy})$$
$$D\nabla^4 w - h(F_{,yy}w_{,xx} + F_{,xx}w_{,yy} - 2F_{,xy}w_{,xy}) + \rho hw_{,tt} = P(x, y, t) \quad (2)$$

where

$$\sigma_x = F_{,yy} \qquad \sigma_y = F_{,xx} \qquad \tau_{xy} = -F_{,xy}$$

are the membrane stresses. When a single mode is assumed and Galerkin's method is applied, the problem reduces to the solution of a single ordinary differential equation in time.

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